Geometric Analysis and General Relativity

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A new point of view on the solutions to the Einstein constraint equations with arbitrary mean curvature

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Abstract: Lying at the center of mathematical physics, the Einstein constraint equations have already been studied intensively especially since an effective version of the conformal method first appeared in the early 1970s. However, the existence and uniqueness of solutions to the constraints were limited to the constant mean curvature (CMC) case through the early 90s, with analogous results for the near-CMC case beginning to appear thereafter. In the last ten years, there has been some limited progress towards the understanding of solutions to the constraints in the far-from-CMC case. Although it was initially conceivable that these far-from-CMC results would lead to a new picture for the non-CMC case that would mirror the good properties of the CMC and near-CMC cases, examples of bifurcations, of non-existence, and of non-uniqueness of solutions have been since discovered. In this talk, I will present a simple approach to construct solutions to the constraints in certain far-from-CMC cases. This finding implies that some known existence results on Yamabe positive manifolds with arbitrary mean curvature can be thought of as perturbations off of the CMC case. This is joint work with Romain Gicquaud.

Stability of black holes and thermodynamics

Stefen Hollands
University of Leipzig

Abstract: I describe the connection between black hole thermodynamics and their (linear) stability. The connection between the two – a priori unconnected – circles of ideas arises through the notion of the “canonical energy". This notion is related on the one hand to the second variation of thermodynamic quantities, and on the other hand to the fluxes of gravitational radiation through horizons and to infinity. I describe how these ideas in connection with methods from geometric analysis can lead to mathematical proofs of (in)stabilities in various situations such as the Gubser-Mitra and Durkee-Reall conjectures, or the stability of higher dimensional black holes under “axi-symmetric" perturbations.

Minimial mass extensions and vacuum stationary

Lan-Hsuan Huang
University of Connecticut

Abstract: Given a compact Riemannian manifold (Ω, g) with boundary, it is of great interest to define an invariant determined by the boundary geometry, so-called the quasi-local mass. Among various proposals, R. Bartnik gave a definition of quasi-local mass in 1989 by considering asymptotically flat extensions of (Ω, g) that satisfy the dominant energy condition and then minimizing the asymptotically defined mass. He proposed the conjecture that an asymptotically flat extension that minimizes the mass (called the minimal mass extension), if exists, must be stationary.

The conjecture has been verified for the special case that the minimization is taken among a smaller class of extensions with nonnegative scalar curvature. In this case, the minimal mass extension must be static, which is by now a well-understood concept. A main step toward Bartnik's stationary conjecture for the general case is to understand the concept of stationary. While stationary is a well-defined spacetime property (i.e. admitting a Killing vector field), it seems to give rise to related but different, sometimes inconsistent, properties of the spatial slices. Only when vacuum the discrepancies disappear. In this talk, we will discuss partial progress toward this conjecture. We identify a general obstruction to promote the dominant energy condition and then use it to show that the minimal mass extension must be vacuum near infinity.

Recent developments on Bartnik's quasi-local mass

Jeff Jauregui
Union College

Abstract: In general relativity, the quasi-local mass problem is to formulate a sensible definition of the mass of a 3-dimensional bounded region Ω in a spacelike hypersurface of a $(3+1)$-dimensional spacetime. In 1989 Robert Bartnik proposed a definition of quasi-local mass given by minimizing the ADM (total) mass among all physically reasonable asymptotically flat extensions of Ω. Bartnik's well-known conjecture is that this infimum is actually attained by some asymptotically flat space, at least under appropriate hypotheses on Ω. We focus on the time-symmetric case, in which "physically reasonable" includes the assumption of nonnegative scalar curvature.

In this talk we will survey some recent results pertaining to Bartnik's definition and the corresponding conjecture. Among these are progress on reconciling the various precise versions of the Bartnik mass; the lower semicontinuity of the ADM mass for Sormani–Wenger intrinsic flat convergence (in joint work with Dan Lee); and counterexamples to the most general version of Bartnik's conjecture (in joint work with Michael Anderson). Since the precise value of the Bartnik mass is known only in specialized cases, we will also discuss work of a number of authors on establishing bounds for the Bartnik mass, as time permits.

Geometric Inequalities for Quasi-Local Masses

Marcus Khuri
Stony Brook University

Abstract: We will describe lower bounds for quasi-local masses in terms of charge, angular momentum, and horizon area. In particular we treat three quasi-local masses based on a Hamiltonian approach, namely the Brown-York, Liu-Yau, and Wang-Yau masses. The geometric inequalities are motivated by analogous results for the ADM mass. They may be interpreted as localized versions of these inequalities, and are also closely tied to the conjectured Bekenstein bounds for entropy of macroscopic bodies. In addition, we give a new proof of the positivity property for the Wang-Yau mass which is used to remove the spin condition in higher dimensions. Furthermore, we generalize a recent result of Lu and Miao to obtain a localized version of the Penrose inequality for the static Wang-Yau mass. This is joint work with A. Alaee and S.-T. Yau.

Variational problem for anisotropic surface energy

Miyuki Koiso
Kyushu University

Abstract: We study variational problems for piecewise-smooth hypersurfaces in the $(n+1)$-dimensional Euclidean space. An anisotropic surface energy is the integral of an energy density that depends on the surface normal over the considered hypersurface, which was introduced to model the surface tension of a small crystal. If the energy density is constant one, the anisotropic surface energy is the usual n-dimensional area of the hypersurface. The minimizer of such an energy among all closed hypersurfaces enclosing the same $(n+1)$-dimensional volume is unique and it is (up to rescaling) so-called the Wulff shape. The Wulff shape and equilibrium hypersurfaces of this energy for volume-preserving variations are not smooth in general. Around each regular (smooth) point, they are graphs of solutions of a second order quasilinear elliptic or hyperbolic partial differential equation. In this talk, by using the " normal cone" at each singular point, we construct a "generalized normal variation" of a considered hypersurface. We then apply this new idea to prove uniqueness of local minimizer of the anisotropic surface energy under several boundary conditions.

Some estimates of mean curvature integrals for convex surfaces

Tatsuya Miura
Tokyo Institute of Technology

Abstract: For a 2-dimensional closed surface Σ immersed into $\mathbb{R}^3$ we consider the following integral of the mean curvature H (defined by the average of the principle curvatures) of power $p \in [1, \infty)$:

$$\mathcal{W}_p(\Sigma) := \int_\Sigma |H|^p.$$

A typical case is that $p = 2$, in which the functional $\mathcal{W}_2$ is nothing but the Willmore energy (or bending energy), and in addition the so-called total mean curvature $\mathcal{W}_1$ is also a classical object. In this talk I will present some lower bounds of the energy $\mathcal{W}_p$ within the framework of convex surfaces, focusing on the “flatness” of a surface. The main motivation is to understand how the boundedness of the Willmore energy prevents convex surfaces collapsing. In fact, building on our estimate, we obtain optimal scaling laws between the Willmore energy and other typical geometric quantities such as the isoperimetric ratio. I will also mention that our argument yields a completely optimal lower bound of the total mean curvature in terms of the diameter.

On the semilinear partial differential equations in homogeneous and isotropic spacetimes

Makoto Nakamura
Yamagata University

Abstract: Several semilinear partial differential equations are considered in homogeneous and isotropic spacetimes. The Cauchy problems of the equations are considered in Lebesgue spaces and Sobolev spaces. Dissipative and anti-dissipative effects from the spatial expansion and contraction on the problems are remarked.

Foliated solutions to Bernoulli's free boundary problem

Michiaki Onodera
Tokyo Institute of Technology

Abstract: Bernoulli's free boundary problem is an overdetermined problem in which one seeks an annular domain such that the capacitary potential satisfies an extra boundary condition. There exist two different types of solutions called elliptic and hyperbolic solutions. Elliptic solutions are "stable" solutions and tractable by variational methods and maximum principles, while hyperbolic solutions are "unstable" solutions of which the qualitative behavior is less known. I will present a joint work with Antoine Henrot (Institut Élie Cartan), in which we introduce a new implicit function theorem based on parabolic maximal regularity, applicable to problems with loss of derivatives. Clarifying the spectral structure of the corresponding linearized operator by harmonic analysis, we prove the existence of foliated hyperbolic solutions as well as elliptic solutions in the same regularity class.

Numerical analysis of discrete total variation flow with manifold constraint

Koya Sakakibara
Kyoto University and RIKEN

Abstract: In this talk, we develop a numerical scheme for a spatially discrete model of total variation flows whose values are constrained to a Riemannian manifold. The difficulty of this problem is that the underlying function space is not convex. Hence, it is hard to calculate a minimizer of the functional with the manifold constraint even if it exists. We overcome this difficulty by " localization technique " using the exponential map, and prove an energy decay and a finite-time error estimate. We also show a few numerical results for the target manifolds are S^2 and $SO(3)$. This talk is based on joint work with Prof. Yoshikazu Giga, Dr. Kazutoshi Taguchi, and Dr. Masaaki Uesaka. The contents of this talk can be found in our preprint [1].

[1] Y. Giga, K. Sakakibara, K. Taguchi, and M. Uesaka. A new numerical scheme for discrete constrained total variation flows and its convergence. arXiv:1904.06105.

The problem of quasi-local mass in general relativity

Richard Schoen
University of California, Irvine

Abstract: This will be a talk about various quasi-local mass quantities which have been useful in general relativity. We will talk about comparisons between different masses. We will also discuss comparison-type theorems for polyhedral surfaces and how these relate to quasi-local masses.

Equality in the logarithmic Sobolev inequality

Asuka Takatsu
Tokyo Metropolitan Universitry

Abstract: It is known that various geometric and functional inequalities (Poincaré inequality, logarithmic Sobolev inequality, isoperimetric inequality and so on) hold on positively curved weighted Riemannian manifolds. Here a weighted Riemannian manifold is a Riemannian manifold equipped with an equivalent probability measure to the Riemannian volume measure. Recently, with the help of the needle decomposition, the rigidity and the stability of these inequalities have been investigated. The needle decomposition is a localization technique established by B. Klartag for reducing a high-dimensional problem to its one-dimensional counterpart. In this talk, I deal with the rigidity problem on the logarithmic Sobolev inequality comparing the relative entropy and the Fisher information.: On a positively curved weighted Riemannian manifold, equality holds in the logarithmic Sobolev inequality if and only if one-dimensional Gaussian space is split off, similar to the rigidity results of Cheng–Zhou on the spectral gap (equivalently, Poincaré inequality) as well as Morgan on the Bakry–Ledoux isoperimetric inequality. In the proof, I show that equality in the logarithmic Sobolev leads to the Poincaré inequality, then the rigidity result in Cheng–Zhou applies. This talk is based on a joint work with Shin-ichi Ohta (Osaka University).

Gradient flows for the harmonic map energy

Peter Topping
University of Warwick

Abstract: The standard L^2 gradient flow for the harmonic map energy is the celebrated harmonic map flow, which is the original geometric flow introduced by Eells and Sampson in 1964. After giving a brief survey of the two-dimensional theory for this flow, we will take a look at the alternative gradient flow for this energy that was introduced jointly with Melanie Rupflin, which evolves both a map and the metric on its domain simultaneously. Now the gradient flow wants to converge to a minimal immersion rather than a harmonic map.

A theory for this flow has been developed over recent years in order to understand its singularity formation and its asymptotics. We will take a look at some of the main points of this theory, including how the flow decomposes a map into minimal immersions. The main focus of the talk will be the recent work joint with Kohout and Rupflin that addresses the question of whether the flow enjoys unique limits or not. We will see when it does and when it does not.