Hyperbolic equations with non analytic coefficients well posed in all Gevrey classes

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In this talk, we consider weakly hyperbolic equations. For the second order equation $\partial_t^2 u = a_2(t)\partial_x^2 u + a_1(t)\partial_x\partial_t u$ with $a_2(t) \in C^k$ (resp. C^∞, resp. $\mathcal{A}$) and $a_1(t) \equiv 0$ under a hyperbolic condition $a_2(t) \geq 0$, [CJS] showed the well-posedness in γ^s for $1 \leq s < 1 + k/2$ (resp. in γ^∞, resp. in C^∞). For the third order equation

$$(P) \qquad \begin{cases} \partial_t^3 u = \{a_3(t)\partial_x^3 + a_2(t)\partial_x^2\partial_t + a_1(t)\partial_x\partial_t^2\}u, \\ \partial_t^h u(0,x) = u_h(x) \quad (0 \leq h \leq 2), \end{cases}$$

we need some additional conditions to get similar kind of results.

1. By [K] we get the wellposedness in γ^∞ of (P) with $a_1(t), a_2(t) \in C^\infty$ and $a_3(t) \equiv 0$ under a hyperbolic condition $a_2(t) \geq 0$.

2. By [S] we get the wellposedness in γ^∞ of (P) with $a_2(t), a_3(t) \in C^\infty$ and $a_1(t) \equiv 0$ under a hyperbolic condition $4a_2(t)^3 - 27a_3(t)^2 \geq {}^\exists c a_3(t)^2$.

3. By [CO] we get the wellposedness in γ^∞ of (P) with $a_1(t), a_3(t) \in C^\infty$ and $a_2(t) \equiv 0$ under a condition

$$\lambda_i^2(t) + \lambda_j^2(t) \leq {}^\exists M \left(\lambda_i(t) - \lambda_j(t)\right)^2, \qquad 1 \leq i < j \leq 3.$$

We denote that

$$\begin{aligned}\Delta(t) = &\prod_{1\leq i<j\leq 3}\left(\lambda_j(t) - \lambda_i(t)\right)^2 \\ =&4a_2(t)^3 - 27a_3(t)^2 + a_1(t)^2a_2(t)^2 - 4a_1(t)^3a_3(t) - 18a_1(t)a_2(t)a_3(t).\end{aligned}$$

When the roots are distict, there exists $c > 0$ such that

$$\Delta(t) \geq c.$$

Remark 1. If the condition in [CO] is satisfied, it follows that

$$\Delta(t) \geq \frac{(\lambda_1(t)^2 + \lambda_2(t)^2)(\lambda_2(t)^2 + \lambda_3(t)^2)(\lambda_3(t)^2 + \lambda_1(t)^2)}{M}.$$

Then we get the following:

Theorem : *If the coefficients are C^k (resp. C^∞, resp. analytic) and satisfy*

$$\left(9a_3(t) + a_1(t)a_2(t)\right)^2 \le {}^\exists M\Delta(t),$$

then the Cauchy problem (P) is well-posed in γ^s for

$$1 \le s < 1 + \frac{k}{4}$$

(resp. in γ^∞, resp. in C^∞).

Remark 2. For the simplicity we suppose that the spacial dimension $n = 1$.

Remark 3. In case when the m-th order equation whose coefficients belong to C^k for $k > 3(m + n + 2)$, [B] proved the wellposedness in γ^s of (P) for

$$1 \le s < 1 + \frac{1}{m-1}.$$

For the proof, we shall improve the theory of quasi-symmetrizers (see [DS]).

References

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