Application of wave packet transform to Schrödinger equations with a subquadratic potential

Keiichi Kato(Tokyo University of Science) *†

January 21, 2012

1 Introduction

In this talk, we consider the following initial value problem of the time dependent Schrödinger equations,

$$\begin{cases} i\partial_t u + \frac{1}{2}\triangle u - V(t,x)u = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $i = \sqrt{-1}$, $u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}$, $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $V(t,x)$ is a real valued function.

We shall determine the wave front sets of solutions to the Schrödinger equations (1) with a sub-quadratic potential $V(t,x)$ by using the representation of the Schrödinger evolution operator of a free particle introduced in [9] via the wave packet transform which is defined by A. Córdoba and C. Fefferman [1]. In particular, we determine the location of all the singularities of the solutions from the information of the initial data. Wave packet transform is called short time Fourier transform in several literatures([7]).

We assume the following assumption on $V(t,x)$.

Assumption 1.1. $V(t,x)$ is a real valued fuction in $C^\infty(\mathbb{R}\times\mathbb{R}^n)$ and there exists $\rho < 2$ such that for all multi-indices α,

$$|\partial_x^\alpha V(t,x)| \le C(1+|x|)^{\rho-|\alpha|}$$

holds for some $C > 0$ and for all $(t,x) \in \mathbb{R}\times\mathbb{R}^n$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. We define the wave packet transform $W_\varphi f(x,\xi)$ of f with the wave packet generated by a function φ as follows:

$$W_\varphi f(x,\xi) = \int_{\mathbb{R}^n} \overline{\varphi(y-x)} f(y) e^{-iy\xi} dy, \quad x,\xi \in \mathbb{R}^n.$$

*This is a joint work with Masaharu Kobayashi and Shingo Ito in Tokyo University of Science.

†email: kato@ma.kagu.tus.ac.jp

In the previous paper [9], we give the representation of the Schrödinger evolution operator of a free particle, which is the following:

$$W_{\varphi(t)}u(t,x,\xi) = e^{-\frac{i}{2}t|\xi|^2} W_{\varphi_0}u_0(x-\xi t,\xi), \tag{2}$$

where $\varphi(t) = \varphi(t,x) = e^{i(t/2)\triangle}\varphi_0(x)$ with $\varphi_0(x) \in \mathcal{S}(\mathbb{R}^n)\backslash\{0\}$ and $W_{\varphi(t)}u(t,x,\xi) = W_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)$. In the following, we often use this convention $W_{\varphi(t)}u(t,x,\xi) = W_{\varphi(t,\cdot)}(u(t,\cdot))(x,\xi)$ for simplicity.

In order to state our results precisely, we prepare several notations. For $\varphi_0(x) \in \mathcal{S}(\mathbb{R}^n)$, we put $\varphi^{(t)}(x) = e^{i(t/2)\triangle}\triangle\varphi_0(x)$ and $\varphi_\lambda^{(t)}(x) = \varphi^{(\lambda t)}(\lambda^{1/2}x)$ for $\lambda \geq 1$. For $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\}$, we call a subset $V = K \times \Gamma$ of $\mathbb{R}^{2n}$ a conic neighborhood of (x_0,ξ_0) if K is a neighborhood of x_0 and Γ is a conic neighborhood of ξ_0 (i.e. $\xi \in \Gamma$ and $\alpha > 0$ implies $\alpha\xi \in \Gamma$). For $\lambda > 0$ and $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, let $x(s;t,x,\lambda\xi)$ and $\xi(s;t,x,\lambda\xi)$ be the solutions to

$$\begin{cases} \dot{x}(s) &= -\xi(s), \quad x(t) = x, \\ \dot{\xi}(s) &= \nabla V(s,x(s)), \quad \xi(t) = \lambda\xi. \end{cases} \tag{3}$$

The following theorem is our main result.

Theorem 1.2. *Let $u_0(x) \in L^2(\mathbb{R}^n)$ and $u(t,x)$ be a solution of* (1) *in $C(\mathbb{R};L^2(\mathbb{R}^n))$. Then under the assumption 1.1, $(x_0,\xi_0) \notin WF(u(t,x))$ if and only if there exists a conic neighborhood $V = K \times \Gamma$ of (x_0,ξ_0) such that for all $N \in \mathbb{N}$ and for all $a \geq 1$ there exists a constant $C_{N,a} > 0$ satisfying*

$$|W_{\varphi_\lambda^{(-t)}}u_0(x(0;t,x,\lambda\xi),\xi(0;t,x,\lambda\xi))| \leq C_{N,a}\lambda^{-N}$$

for $\lambda \geq 1$, $a^{-1} \leq |\xi| \leq a$ and $(x,\xi) \in V$.

Remark 1.3. K. Yajima [15] shows that the fundamental solution is not smooth anywhere with respect to x for $t > 0$ in one space dimension, if $V(t,x)$ is super quadratic.

Remark 1.4. $W_{\varphi_\lambda^{(-t)}}u_0(x,\xi)$ is the wave packet transform of $u_0(x)$ with a window function $\varphi_\lambda^{(-t)}(x)$.

Remark 1.5. In [10], the authors investigate the wave front sets of solutions to Schrödinger equations of a free particle and a harmonic oscillator via the wave packet transformation.

Corollary 1.6. *If $\rho < 1$, then $(x_0,\xi_0) \notin WF(u(t,x))$ if and only if there exists a conic neighborhood $V = K \times \Gamma$ of (x_0,ξ_0) such that for all $N \in \mathbb{N}$ and for all $a \geq 1$ there exists a constant $C_{N,a} > 0$ satisfying*

$$|W_{\varphi_\lambda^{(-t)}}u_0(x-\lambda t\xi,\lambda\xi)| \leq C_{N,a}\lambda^{-N}$$

for $\lambda \geq 1$, $a^{-1} \leq |\xi| \leq a$ and $(x,\xi) \in V$.

W. Craig, T. Kappeler and W. Strauss [2] have treated this smoothing property microlocally. They have shown for a solution of (1) that for a point $x_0 \neq 0$ and a conic neighborhood Γ of x_0, $\langle x\rangle^r u_0(x) \in L^2(\Gamma)$ implies $\langle \xi\rangle^r \hat{u}(t,\xi) \in L^2(\Gamma')$ for a conic neighborhood of Γ' of x_0 and for $t \neq 0$, though they have considered more general operators. Several mathematicians have shown this kind of results for Schrödinger operators [4], [5], [11], [13], [14].

A. Hassell and J. Wunsch [8] and S. Nakamura [12] determine the wave front set of the solution by means of the initial data. Hassell and Wunsch have studied the singularities by using “scattering wave front set”. Nakamura has treated the problem in semi-classical way. He has shown that for a solution $u(t,x)$ of (1) and $h > 0$ $(x_0,\xi_0) \notin WF(u(t))$ if and only if there exists a C_0^∞ function $a(x,\xi)$ in $\mathbb{R}^{2n}$ with $a(x_0,\xi_0) \neq 0$ such that $\|a(x+tD_x, hD_x)u_0\| = O(h^\infty)$ as $h \downarrow 0$. On the other hand, we use the wave packet transform instead of the pseudo-differential operators.

2 Outline of the proof of Theorem 1.2

In order to prove Theorem 1.2, we introduce the definition of wave front set $WF(u)$ and the characterization of wave front set by G. B. Folland [6].

Definition 2.1 (Wave front set)**.** For $f \in \mathcal{S}'(\mathbb{R}^n)$, we say $(x_0,\xi_0) \notin WF(f)$ if there exist a function $a(x)$ in $C_0^\infty(\mathbb{R}^n)$ with $a(x_0) \neq 0$ and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ there exists a positive constant C_N satisfying

$$|\widehat{af}(\xi)| \leq C_N(1+|\xi|)^{-N}$$

for all $\xi \in \Gamma$.

To prove Theorem 1.2, we use the following characterization of the wave front set by G. B. Folland [6]. Let $\varphi \in \mathcal{S}$ satisfying $\int x^\alpha \varphi(x)dx \neq 0$ for some multi-index α. For fixed b with $0 < b < 1$, we put $\varphi_\lambda(x) = \lambda^{nb/2}\varphi(\lambda^b x)$.

Proposition 2.2 (G. B. Folland [6, Theorem 3.22] and T. Ōkaji [13, Theorem2.2])**.** *For $f \in \mathcal{S}'(\mathbb{R}^n)$, we have $(x_0,\xi_0) \notin WF(f)$ if and only if there exist a conic neighborhood K of x_0 and a conic neighborhood Γ of ξ_0 such that for all $N \in \mathbb{N}$ and for all $a \geq 1$ there exists a constant $C_{N,a} > 0$ satisfying*

$$|W_{\varphi_\lambda} f(x,\lambda\xi)| \leq C_{N,a}\lambda^{-N}$$

for $\lambda \geq 1$, $a^{-1} \leq |\xi| \leq a$, $x \in K$ and $\xi \in \Gamma$.

Remark 2.3. Folland [6] has shown that the conclusion follows if the window function φ is an even and nonzero function in $\mathcal{S}(\mathbb{R}^n)$ and $b = 1/2$. In Ōkaji [13], the proof of Proposition 2.2 for $b = 1/2$ is given.

Remark 2.4. Folland [6] and Ōkaji [13] have proved for $b = 1/2$. It is easy to extend for $0 < b < 1$.

Outroof of Theorem 1.2. The initial value problem (1) is transformed by the wave packet transform to

$$\begin{cases} \left(i\partial_t + i\xi \cdot \nabla_x - i\nabla_x V(t,x) \cdot \nabla_\xi - \frac{1}{2}|\xi|^2 - \widetilde{V}(t,x)\right) \times \\ \qquad\qquad\qquad\qquad W_{\varphi(t)}u(t,x,\xi) = Ru(t,x,\xi), \\ W_{\varphi(0)}u(0,x,\xi) = W_{\varphi_0}u_0(x,\xi), \end{cases} \tag{4}$$

where $\widetilde{V}(t,x) = V(t,x) - \nabla_x V(t,x) \cdot x$ and

$$Ru(t,x,\xi) = \sum_{j,k} \int \overline{\varphi(y-x)} V_{jk}(t,x,y)(y_j - x_j)(y_k - x_k)u(t,y)e^{-i\xi y}dy$$

with $V_{jk}(t,x,y) = \int_0^1 \partial_j \partial_k V(t, x + \theta(y-x))(1-\theta)d\theta$. Solving (4), we have the integral equation

$$\begin{aligned} W_{\varphi(t)}u(t,x,\xi) &= e^{-i\int_0^t \{\frac{1}{2}|\xi(s;t,x,\xi)|^2 + \widetilde{V}(s,x(s;t,x,\xi))\}ds} W_{\varphi_0}u_0(x(0;t,x,\xi), \xi(0;t,x,\xi)) \\ &\quad - i\int_0^t e^{-i\int_s^t \{\frac{1}{2}|\xi(s_1;t,x,\xi)|^2 + \widetilde{V}(s_1,x(s_1;t,x,\xi))\}ds_1} Ru(s, x(s;t,x,\xi), \xi(s;t,x,\xi))ds, \end{aligned}$$

where $x(s;t,x,\xi)$ and $\xi(s;t,x,\xi)$ are the solutions of

$$\begin{cases} \dot{x}(s) &= \xi(s), \ x(t) = x, \\ \dot{\xi}(s) &= -\nabla_x V(s, x(s)), \ \xi(t) = \xi. \end{cases}$$

For fixed t_0, we have

$$\begin{aligned} & W_{\varphi_\lambda^{(-t_0)}(t)} u(t, x(t;t_0,x,\lambda\xi), \xi(t;t_0,x,\lambda\xi)) \\ &\qquad = e^{-i\int_0^t \{\frac{1}{2}|\xi(s;t_0,x,\lambda\xi)|^2 + \widetilde{V}(s,x(s;t_0,x,\lambda\xi))\}ds} W_{\varphi_\lambda^{(-t_0)}} u_0(x(0;t_0,x,\lambda\xi), \xi(0;t_0,x,\lambda\xi)) \\ &\quad + i\int_0^t e^{-i\int_s^t \{\frac{1}{2}|\xi(s_1,t_0,x,\lambda\xi)|^2 + \widetilde{V}(s_1,x(s_1;t_0,x,\lambda\xi))\}ds_1} Ru(s, x(s;t_0,x,\lambda\xi), \xi(s;t_0,x,\lambda\xi))ds, \end{aligned} \tag{5}$$

substituting $(x(t;t_0,x,\lambda\xi), \xi(t;t_0,x,\lambda\xi))$ and $\varphi_\lambda^{(-t_0)}(x)$ for (x,ξ) and $\varphi_0(x)$ respectively.

In order to prove the equivalence of the statements in Theorem 1.2, for some neighborhood K of x_0 and some conic neighborhood Γ of ξ_0, it suffices to show that the following assersion $P(\sigma) = P(\sigma, \varphi_0, K, \Gamma)$ holds for all $\sigma \geq 0$ and for all $\varphi_0 \in \mathcal{S}$ satisfying

$\int x^\alpha \varphi_0(x)dx \neq 0$ for some α under the condition that $(x_0, \xi_0) \notin WF(u(t,x))$.
$P(\sigma) = P(\sigma, \varphi_0, K, \Gamma)$: "There exists a constant $C_{\delta,a} > 0$ such that

$$|W_{\varphi_\lambda^{(-t_0)}(t)} u_0(x(0;t_0,x,\lambda\xi), \xi(0;t_0,x,\lambda\xi))| \leq C_{N,a}\lambda^{-\sigma}$$

for all $x \in K$, all $\xi \in \Gamma$ with $1/a \leq |\xi| \leq a$, all $\lambda \geq 1$ and $0 \leq t \leq t_0$. "

We put $b = \min((2-\rho)/2, 1/2)$. Under the condition $(x_0, \xi_0) \notin WF(u(t,x))$, we show that

(1) $P(0)$ holds for all $\varphi_0 \in \mathcal{S}$ satisfying $\int x^\alpha \varphi_0(x)dx \neq 0$ for some α.

(2) For all $\sigma \geq 0$, $P(\sigma + b)$ holds for all $\varphi_0 \in \mathcal{S}$ satisfying $\int x^\alpha \varphi_0(x)dx \neq 0$ for some α under the assumption that $P(\sigma)$ holds.

$\square$

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